Bead on a rotating hoop : a simple yet feature-rich dynamical system

Introduction of Damping

and

Bifurcation Analysis

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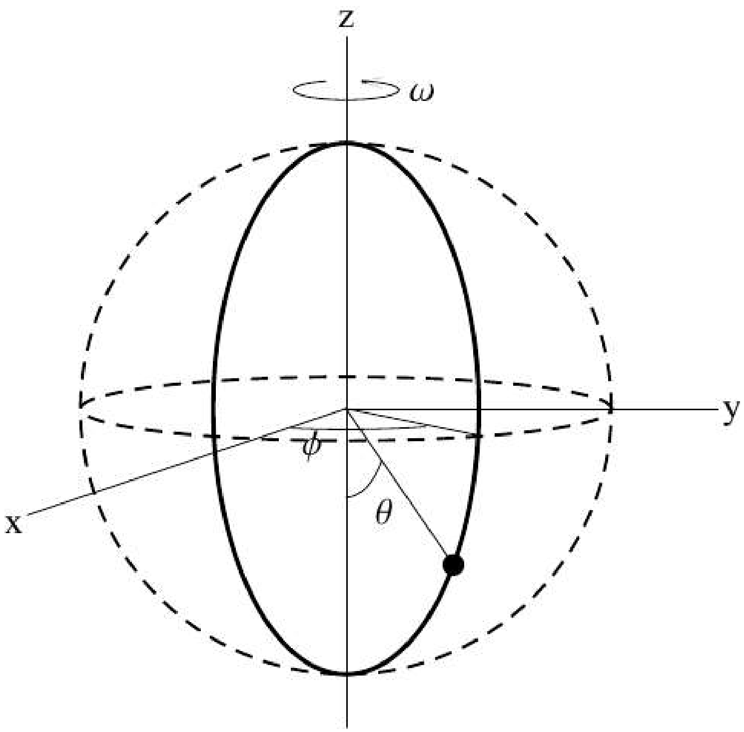
The Physical System



- Lagrangian of the system:

$$L(\theta,\dot{\theta}) = (ma^2/2)\left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right) + mga\cos\theta$$

- Euler-Lagrange equation (in dimensionless form):

$$\theta'' = -\sin\theta(1 - k\cos\theta)$$

where $k = \omega^2/\omega_c^2$; $\omega_c = \sqrt{g/a}$

$$\theta' \equiv d\theta/d\tau \ ; \ \ \tau = \omega_c t$$

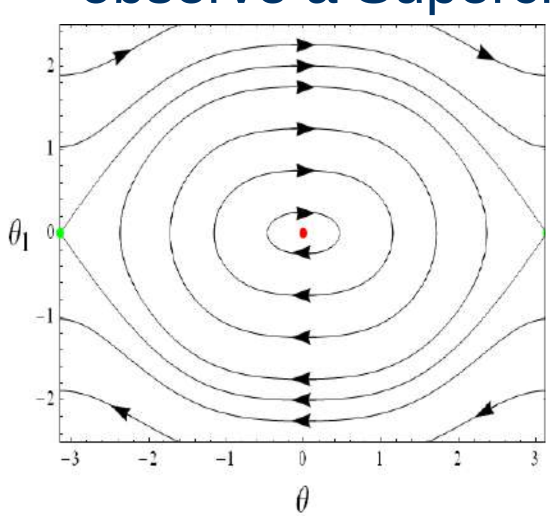
- When $\omega = 0$, this reduces to

$$\theta'' = -\sin\theta,$$

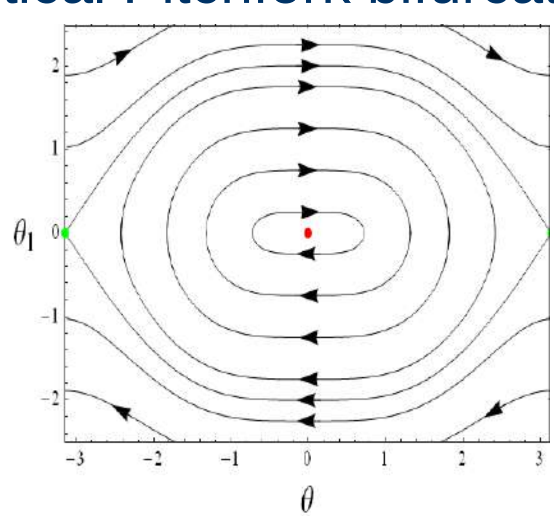
which is the simple pendulum.

Supercritical Pitchfork Bifurcation

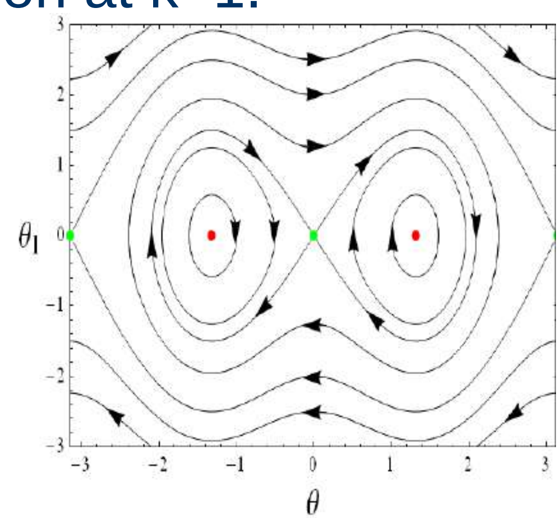
- For k ≤ 1, the bottom of the hoop (ϴ=0) is a stable equilibrium and the top (ϴ=π) is an unstable equilibrium.
- For k > 1, ϴ=0 becomes unstable and new stable equilibrium points branch out in opposite directions at $\theta = \pm\cos^{-1}(1/k)$. We observe a Supercritical Pitchfork bifurcation at k=1.



Center, k = 0.75



Weak center, k = 1



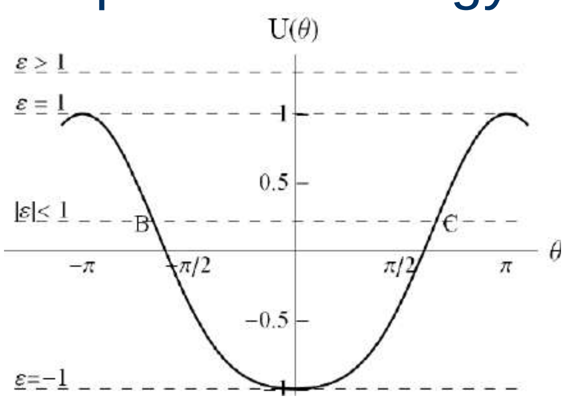
Saddle, k = 4

Effective Potential Energy

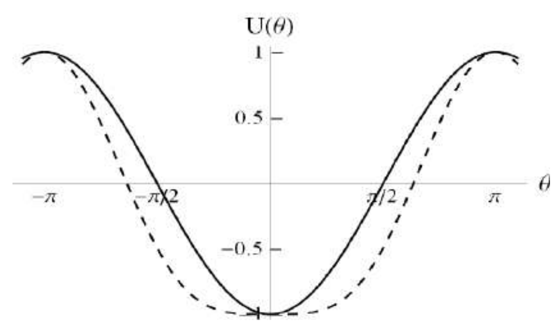
- We can identify a conserved quantity,

$$\varepsilon = \theta'^2/2 - (k/2)\sin^2\theta - \cos\theta$$

- which may be called the effective energy. The effective potential energy is $U(\theta) = -(k/2)\sin^2\theta - \cos\theta$.

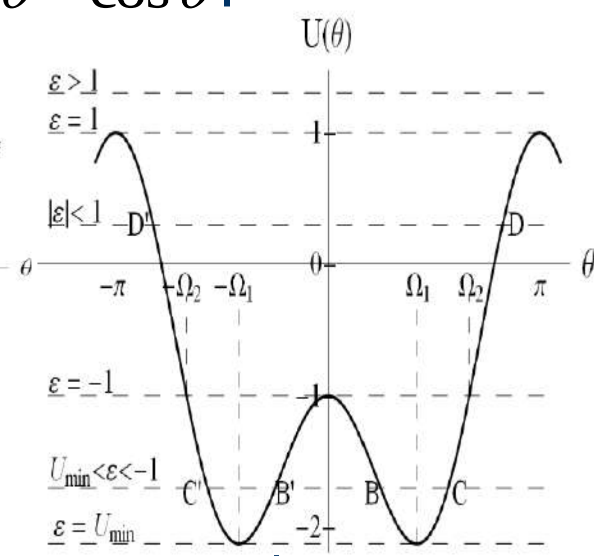


k < 1



solid: k = 0.01

dashed: k = 1



k > 1

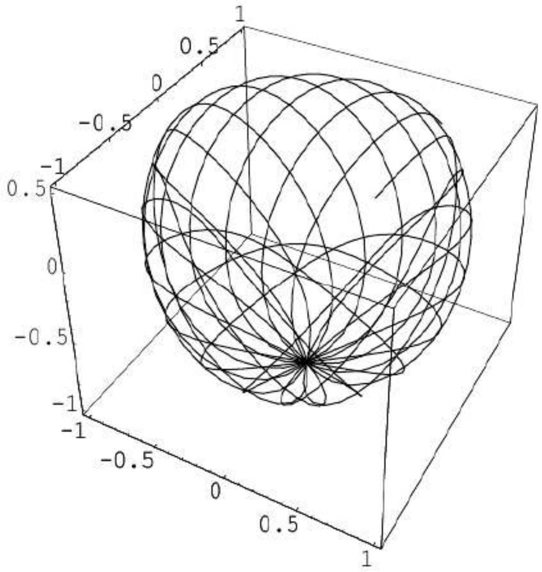
Critical Slowing Down

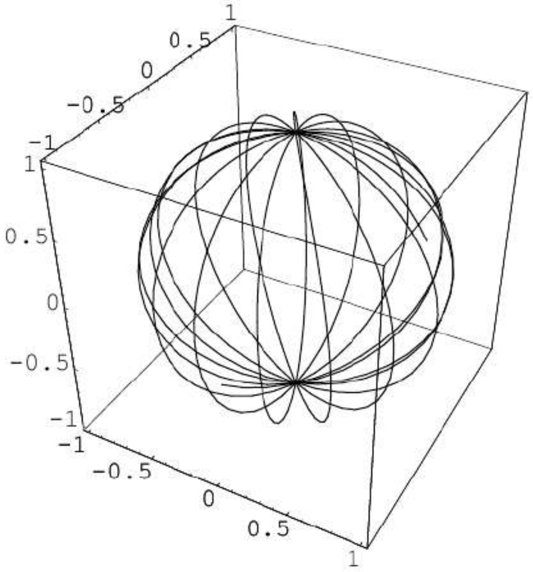
- The potential is much flatter at $\theta = 0$ for k=1 than for k<1, resulting in a very weak center.
- Critical slowing down occurs: solutions decay very slowly. Time period for a given amplitude is much larger for k=1.

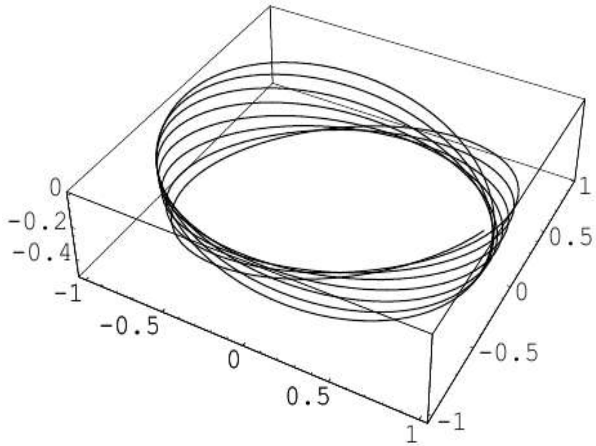
Time period	k = 0	k = 0.5	k = 0.75	k = 1
Ampl = π/10	6.32	8.78	12.00	33.71
Ampl = π/20	6.29	8.86	12.41	66.93
Ampl = π/40	6.29	8.88	12.53	133.62
Ampl = π/80	6.28	8.88	12.55	267.12

- For small amplitudes, time period is nearly inversely proportional for k = 1.

Some Trajectories







Oscillation about the bottom for k<1

Whirling motion

Oscillation about $\theta = \cos^{-1}(1/k)$ for k>1

Introduction of Damping

- A friction term is included in the governing equation, yielding,

$$\theta'' = -\sin\theta(1 - k\cos\theta) - \mu\theta'$$

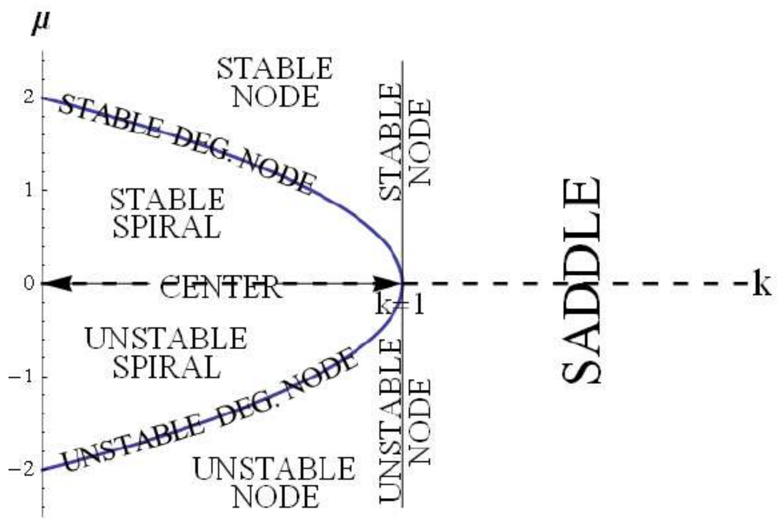
- For phase plane analysis, this is written as,

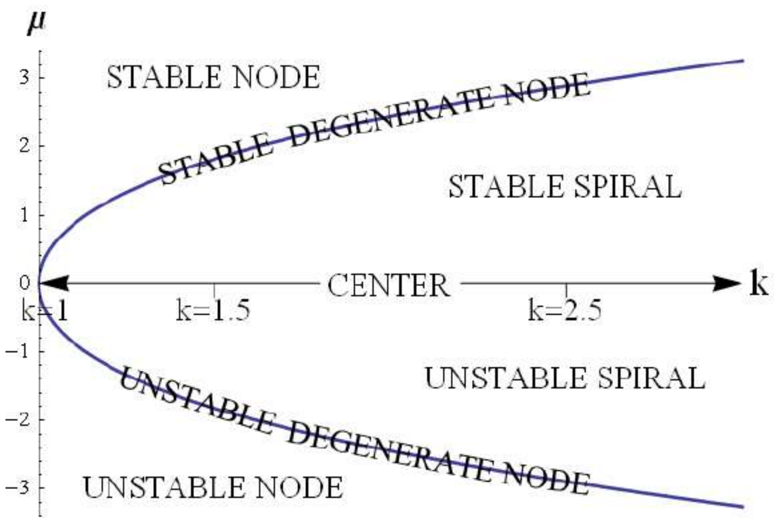
$$\theta' = \theta_1$$
$$\theta_1' = -\sin\theta(1 - k\cos\theta) - \mu\theta_1$$

- This is invariant under $\theta \to -\theta$, $\theta_1 \to -\theta_1$. So, alternate quadrants have similar trajectories.
- This is also invariant under $\theta_1 \to -\theta_1$, $\tau \to -\tau$, $\mu \to -\mu$, implying that phase portraits for –ve damping are reflections of +ve damping phase portraits with arrows reversed. So, stable fixed points will change to unstable counterparts for negative damping.

Nature of Fixed Points

- Introduction of friction doesn't change location of fixed points, but does change their nature.



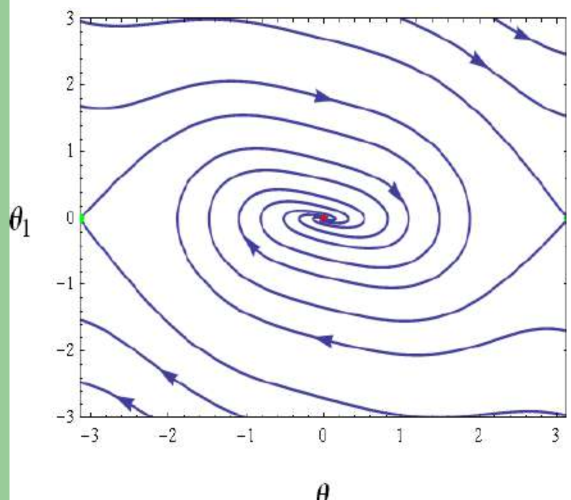


Nature of (0,0) Nature of $(\pm\cos^{-1}(1/k),0)$

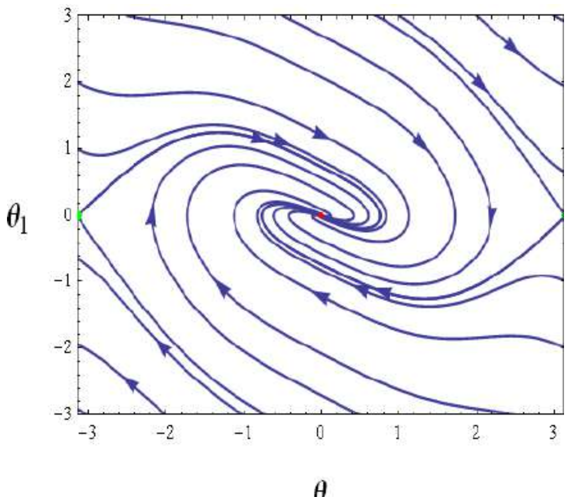
- $(\pm\pi,0)$ remains a saddle for $k \geq 0$ for all values of μ.

Spiral-Node Bifurcation for k < 1:

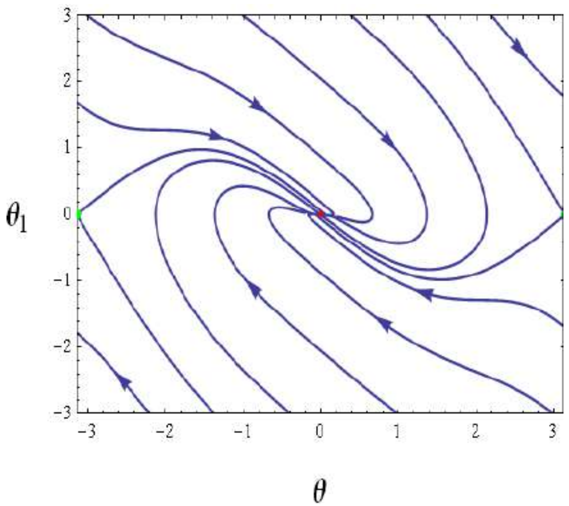
- As damping is increased from 0 keeping k < 1, the nature of oscillation about $\theta = 0$ changes from undamped to under-damped to critically damped to overdamped.
- The fixed point at (0,0) changes from a stable spiral to stable node at $\mu = 2\sqrt{1-k}$, showing a spiral-node bifurcation.



$k = 0.91, \mu = 0.3$



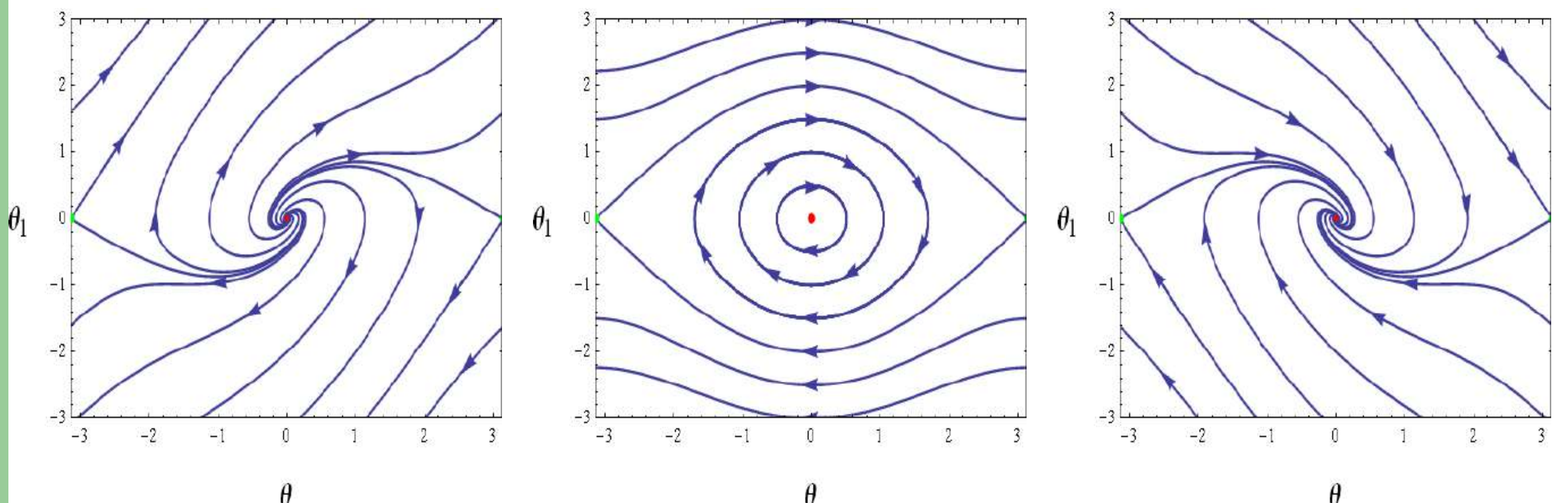
$k = 0.91, \mu = 0.6$



$k = 0.91, \mu = 1$

Degenerate Hopf Bifurcation for k<1

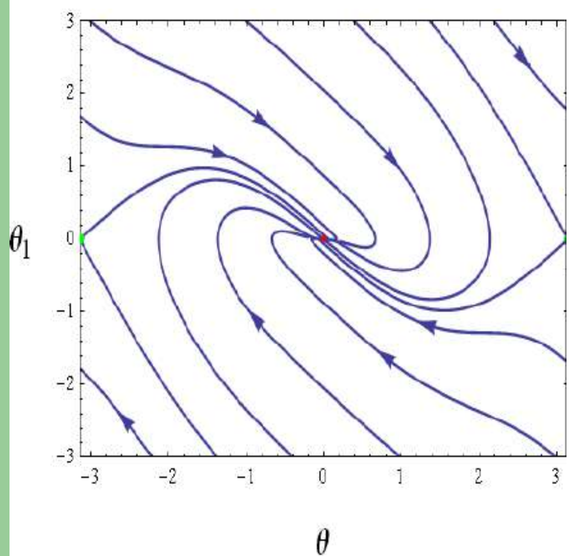
- For k < 1, as the damping coefficient μ is varied through 0, (0,0) changes from an unstable spiral for $\mu < 0$ to a center at $\mu = 0$ to a stable spiral for $\mu > 0$, exhibiting a degenerate Hopf bifurcation.



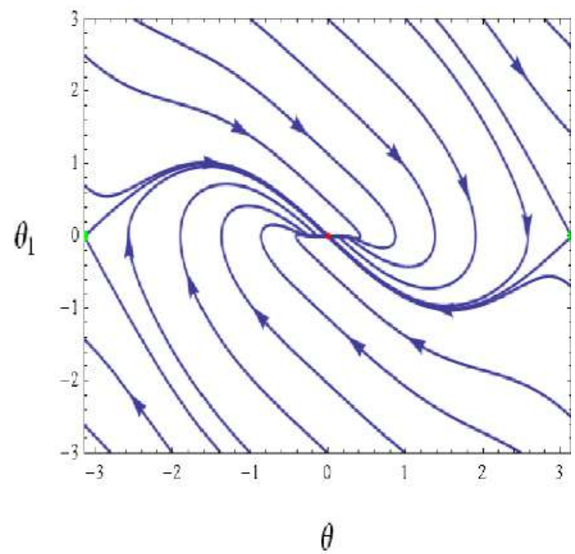
$k=0, \mu=-1$ $\quad$ $k=0, \mu=0$ $\quad$ $k=0, \mu=1$

Supercritical Pitchfork Bifurcation for nonzero Damping

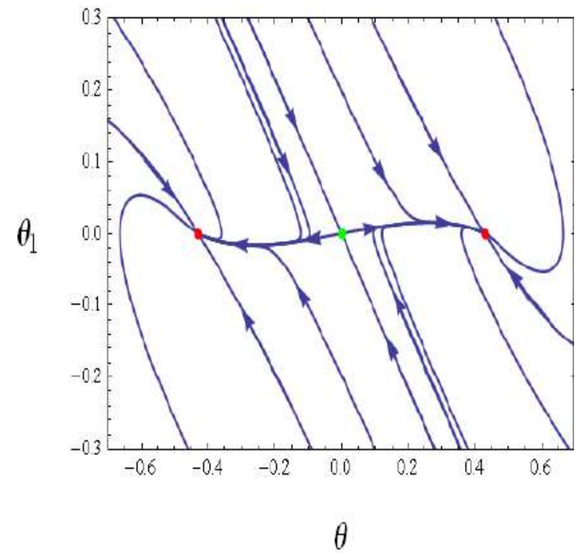
- For +ve damping, as k is increased beyond 1, (0,0) transforms from a stable node to a saddle. Two new stable nodes appear at $\theta = \pm\cos^{-1}(1/k)$ and branch out in opposite directions. Thus, a supercritical pitchfork bifurcation occurs at k=1 (shown for μ=1)



Stable node k=0.91



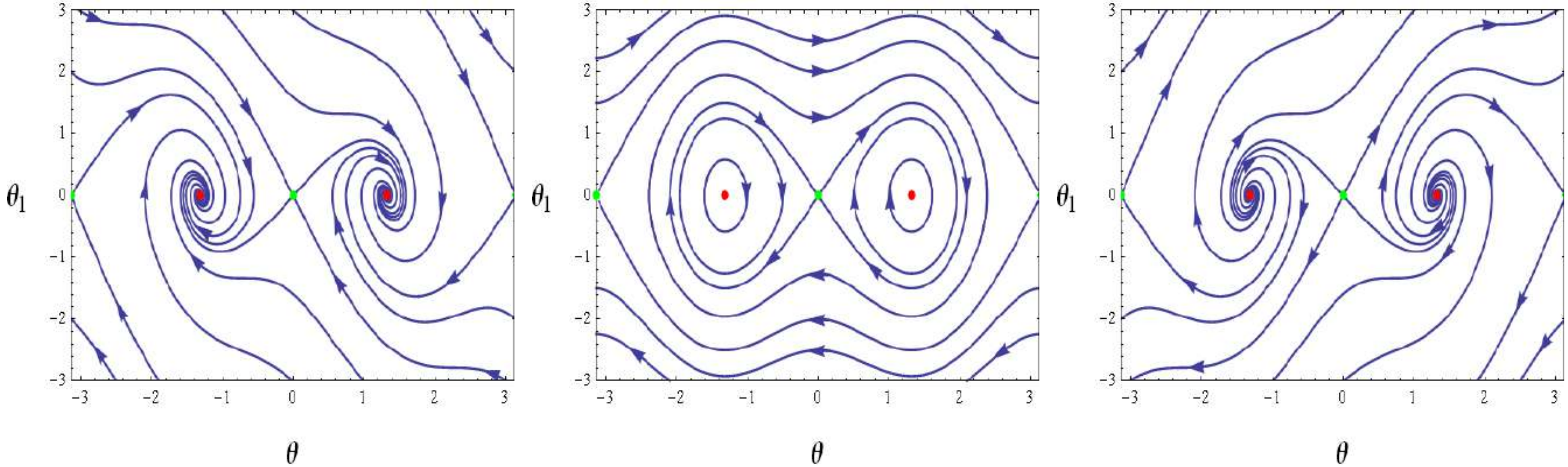
Stable node k=1



Saddle k=1.1

Hopf Bifurcation for k $\geq$ 1

- We get a degenerate Hopf bifurcation as μ is swept through -ve to +ve values, keeping k>1. The fixed point $(\pm\cos^{-1}(1/k),0)$ turns into a stable spiral from an unstable spiral (shown for k=4).

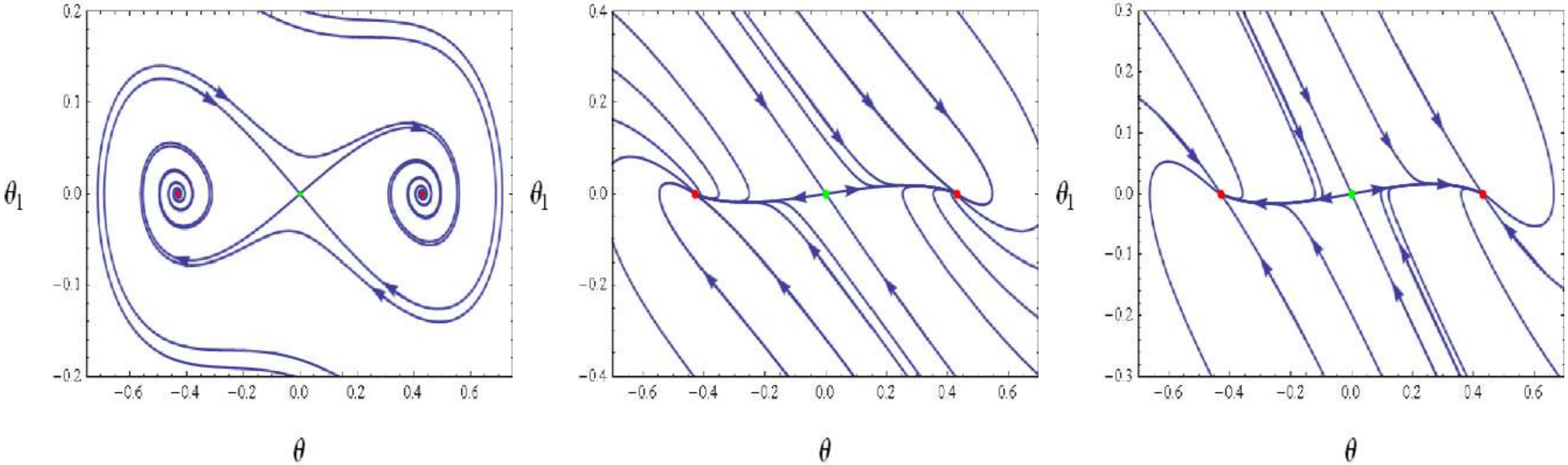


Unstable spiral, μ=-1 Center, μ=0 Stable spiral, μ=1

- Hopf bifurcation is also observed for k=1 as μ is varied through 0. Here (0,0) changes from an unstable node to a stable node.

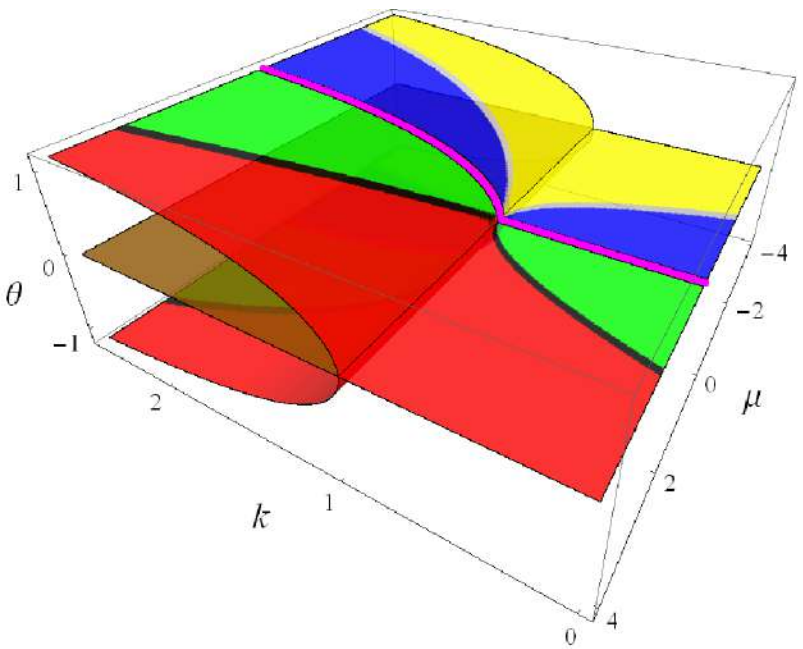
Spiral-Node Bifurcation for k > 1

- For k>1, $(\cos^{-1}(1/k), 0)$ is a spiral for $0 < |\mu| < 2\sqrt{k - 1/k}$. It becomes a degenerate node at $|\mu| = \mu_c \equiv 2\sqrt{k - 1/k}$, and finally a node as $|\mu|$ is raised further. Thus, we get a spiral-node bifurcation (shown below for k=1.1).



Stable spiral, μ=0.1 Stable deg. node, $\mu = \mu_c$ Stable node, μ=1

The Bifurcation Diagram



The Bifurcation Diagram

- Bifurcations occur across the boundaries.

Red → stable node, Green → stable spiral, Blue → unstable spiral, Yellow → unstable node Brown → saddle, Pink→center, Gray → unstable deg. node, Black → stable deg. node

- At the borders, exact nature of fixed points can be derived by changing to polar coordinates and using order of magnitude arguments [arXiv:1201.1218].

The Bifurcation Table

The Bifurcation Table

Points in k-μ space	Bifurcation along k axis	Bifurcation along μ axis
1) (k, 0) ; k $\neq$ 1	--	Degenerate Hopf
2) (k, $\pm 2\sqrt{1-k}$) ; 0 $\leq$ k < 1	Spiral-to-Node	Spiral-to-Node
3) (1, 0)	Supercritical Pitchfork	Hopf
4) (1, μ) ; $\mu \neq 0$	Supercritical Pitchfork	--
5) (k, $\pm 2\sqrt{k-1/k}$) ; k > 1	Spiral-to-Node	Spiral-to-Node

- Table shows only those bifurcations resulting as a variation of either k or μ. Traversing suitable curves in parameter space, one can move from one region to another region of different dynamics, yielding new kinds of bifurcations. The physical relevance of such bifurcations is subject to further inquiry.

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Questions?