1. Should you have all-nighters?

It is approaching midnight on Monday, and you have a project report due at noon on Tuesday. So, you want to get the maximum amount of work done in the next 12 hours. But you are feeling sleepy! How many hours should you sleep?

Let's find out. For simplicity, we'll assume that you either work or sleep!

Here's what we know: if you sleep for x number of hours, then for the next x hours, you can work at your peak rate, which is 1 unit/hour. For the remaining $(12-2x)$ hours, your rate of work falls exponentially with time as $e^{-t/2}$. Therefore, the total amount of work you get done in 12 hours is

$$w(x) = x + \int_0^{12-2x} e^{-t/2}\, dt \ .$$

(a) Evaluate the integral to find $w(x)$ as a function of x.

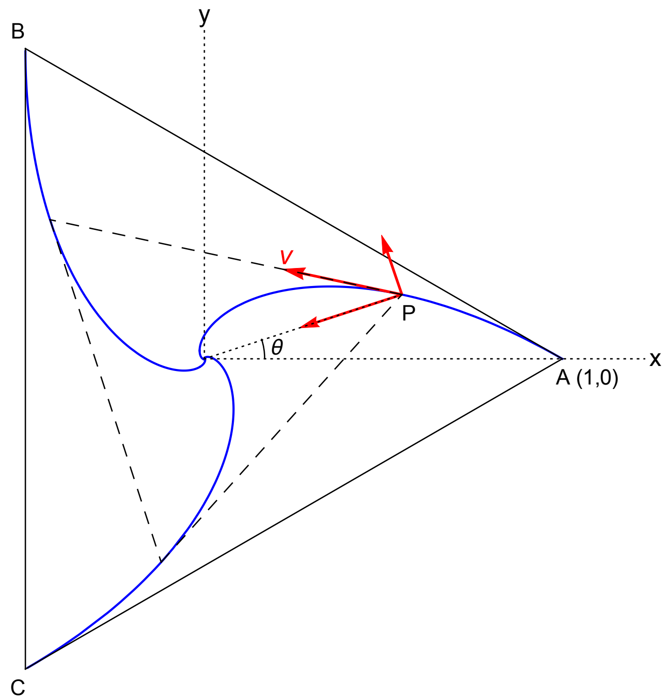
(b) For what value of x is $w(x)$ maximum, and what is the maximum value of $w(x)$? Round your answers to one decimal place.

(c) How many units of work can you do if you opt for an all-nighter, i.e., $x = 0$? Compare this with your answer in part (b)!

(d) Expand $w(x)$ in a Taylor series about the maximum point. Evaluate terms up to second order.

(e) Evaluate $w(x)$ and its Taylor expansion at $x = 6$. Round your answers to one decimal place.

2. Followers of each other



Consider again the problem we discussed in section. Three particles are initially located at the three corners of an equilateral triangle ABC. They start moving toward the particle at the next corner (counterclockwise). At any time during the journey, each particle is moving toward its neighbor at a constant speed v. We found out in section that they will finally meet at the center, as shown in the figure.

Consider the instant when the particle starting from A(1,0) has reached point P(x, y).

(i) Find out the radial and tangential components of its velocity, as indicated in the figure.

(ii) The radial component causes r to <u>decrease</u>, and the tangential component causes θ to <u>increase</u>, where r and θ are the usual polar co-ordinates. Find out $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ <u>in terms of</u> r and v.

Hint : The tangential component equals $r\frac{d\theta}{dt}$ in magnitude.

(iii) Find out the x and y components of the velocity <u>in terms of</u> θ and v.

Hint: You can take either of the following approaches: (a) use $x = r\cos\theta$ and $y = r\sin\theta$, take derivatives, and substitute your result from (ii). Or, (b) directly find the components using geometry.

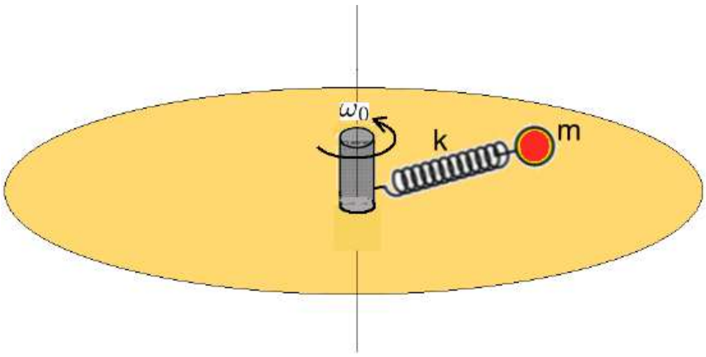
(iv) Find out the x and y components of the acceleration <u>in terms of</u> r, θ, and v.

(v) Find the magnitude of the net acceleration <u>in terms of</u> v and r. Draw its direction at point P in the figure.

Hint: Don't do any calculation to find the direction!

(vi) Using your results in (ii), find $\frac{dr}{d\theta}$ in terms of r. <u>Guess</u> how many times the particle spirals around the origin before reaching there.

3. Spring-mass system on a rotating platform



Mass m is attached to a massless spring which is fixed to a shaft rotating at a constant angular speed ω_0 in a horizontal plane. The spring has an un-stretched length l_0 and spring constant k. It cannot bend sideways. Ignore effects of air drag, friction, or gravity.

Our goal is to determine how the radial distance of mass m from the shaft, namely r, changes with time. Does it oscillate, increase indefinitely, or shrink to zero?

(a) Write down the equation of motion in the radial direction. Cast your equation into the form $\ddot{r} = Ar + B$, where A and B are constants.

(b) Solve your equation to find $r(t)$ when $\omega_0^2 = k/m$. Assume the initial conditions $r(t=0) = l_0$ and $\dot{r}(t=0) = v_0$.

(c) Now consider what happens for $\omega_0^2 < k/m$. Solve your equation in (a) to find $r(t)$. Assume the initial conditions $r(t=0) = -B/A$ and $\dot{r}(t=0) = v_0$.

Hint: It might help to re-write your equation in terms of the variable $R = r + B/A$.

(d) Simplify the solution you found for $r(t)$ in (c) for $\omega_0 = 0$. Does the result make sense?

(e) Now consider $\omega_0^2 > k/m$. Solve your equation in (a) to find $r(t)$ for the initial conditions $r(t=0) = -B/A$ and $\dot{r}(t=0) = v_0$. Does $r(t)$ oscillate?

4. Momentum and energy: conserved? invariant?

A. Elastic collision. Consider the figure below: a 1 kg mass moving with a speed of 3 m/s collides elastically with a 2 kg mass at rest. The collision takes place in a frictionless horizontal plane.

(a) Find out the velocity of each mass after the collision, and show them as arrows in the above figure.

(b) Let's view the same collision from a frame moving to the right at a constant speed of 1 m/s. Using your results in (a), draw the velocities of each mass before and after the collision, *as seen from the moving frame*, in the figure below.

(c) Which of the following options is correct for the above collision, *as seen from the moving frame?*

(i) Both the total momentum and the total kinetic energy of the two masses are conserved.

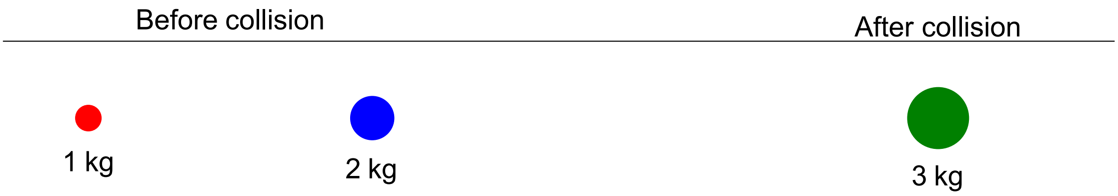
(ii) The total momentum is conserved, but the total kinetic energy is not.

(iii) The total kinetic energy is conserved, but the total momentum is not.

(iv) Neither the total momentum nor the total kinetic energy is conserved.

B. Inelastic collision. Now consider an inelastic collision where the two masses stick together, and move as a single body after the collision (see figure below).

(a) Draw the velocities of each mass before and after the collision in the figure below, *as seen from a moving frame which is traveling to the right at 1 m/s.*



(b) Which of the following options is/are correct for the above inelastic collision?

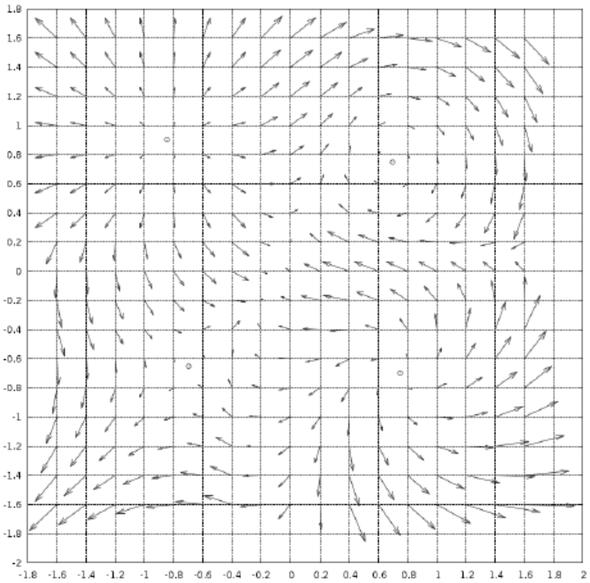
(i) The total momentum before the collision in the ground frame is the same as that in the moving frame.

(ii) The total momentum is conserved during the collision, both in the ground frame and in the moving frame.

(iii) The total kinetic energy before the collision in the ground frame is the same as that in the moving frame.

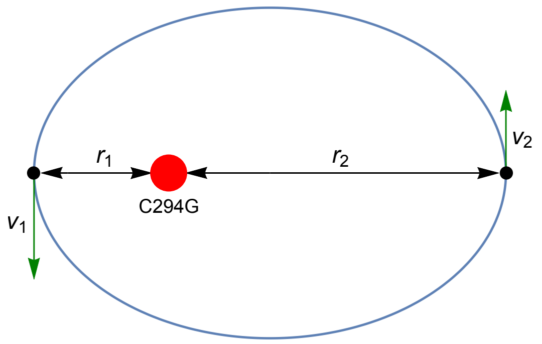
(iv) The total kinetic energy lost during the collision in the ground frame is the same as that in the moving frame.

5. Conservative and nonconservative forces



A. The arrows in the figure represent the magnitude and direction of a force field. Is the force conservative or non-conservative? Give brief reasoning.

B. Suppose you discover a new planetary system : planet PH116YS is revolving around the star C294G in an elliptical orbit! You measure that the planet is moving with speed $v_1 = 6 \times 10^4$ m/s when it is nearest to the star, $r_1 = 10^{11}$ m, and with speed $v_2 = 3 \times 10^4$ m/s when it is farthest, $r_2 = 2 \times 10^{11}$ m. Use your data to calculate the mass of C294G. Take $G = 7 \times 10^{-11}$ N-m^2/kg^2.



[The gravitational potential energy of two bodies is given by $-GMm/r$.]

C. Suppose a particle is experiencing a conservative force $\vec{F}(x)$ for which we can define a potential energy $U(x)$. Which of the following is/are possible for a given location, $x = a$?

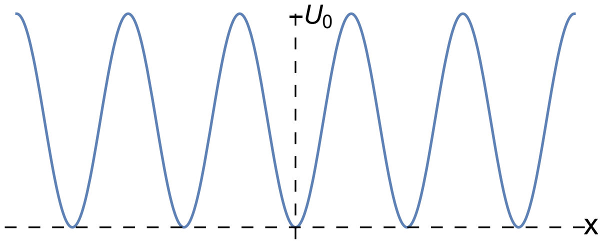
(i) $\vec{F} = \vec{0}$, $U = 0$

(ii) $\vec{F} \neq \vec{0}$, $U = 0$

(iii) $\vec{F} = \vec{0}$, $U \neq 0$

(iv) $\vec{F} \neq \vec{0}$, $U \neq 0$

D. Cold-atom labs routinely use laser beams to trap atoms in optical lattices. Here each atom interacts with light, which can be modeled by a potential energy profile varying periodically in space, $U(x) = U_0 \sin^2(2\pi x/\lambda)$ (see figure). Here λ is the laser wavelength, and U_0 is proportional to the laser intensity.

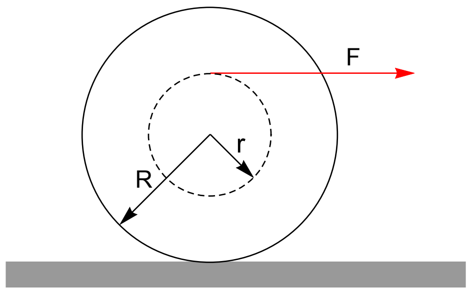


Consider an atom of mass m trapped near the potential energy minimum at $x = 0$. Find out its frequency of small oscillations about the minimum, in terms of U_0, λ, m.

[Turns out atoms do not follow the rules of classical mechanics. However, their behavior is still governed by the frequency you calculated. In particular, it determines their possible states of motion (wave-functions and energies). Notice how you can control them by changing the laser intensity / wavelength.]

6. Do you need friction to roll?

A. (i) A spool rests on a *frictionless* surface. A thread wound on the spool is pulled horizontally with force F, as shown in the figure. For what value of the ratio r/R will the spool roll without slipping? You can approximate the moment of inertia of the spool (about its axis) to be $\frac{1}{2}MR^2$.



(ii) What happens if the thread is pulled horizontally from the bottom instead of from the top? What if we have friction? Which way does the spool roll? Careful!

B. A cubical block of mass m and edge a slides down a rough inclined plane of inclination θ with a *constant velocity*. The torque of the normal force on the block about its center has a magnitude

(i) zero.

(ii) Mga

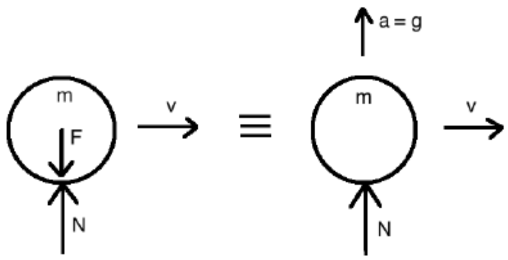
(iii) $Mga \sin\theta$

(iv) $Mga \sin\theta/2$.

7. Does a moving object weigh more?

Consider a ball of rest mass m moving with speed v on the balance. What weight does the balance read? Is it mg? or γmg?

We can answer the question using the equivalence principle, which says that the reading should be identical to the situation where no gravity is present, but the balance is accelerating upward at a rate g.



Let $+x$ denote the direction to the right, and $+z$ denote the vertical upward direction. Then, the force equation along z is given by

$$N = \frac{dp_z}{dt} = \frac{d}{dt}\left(\frac{mv_z}{\sqrt{1-\frac{v_x^2+v_z^2}{c^2}}}\right) = \frac{d}{dv_z}\left(\frac{mv_z}{\sqrt{1-\frac{v_x^2+v_z^2}{c^2}}}\right)\frac{dv_z}{dt} = g\frac{d}{dv_z}\left(\frac{mv_z}{\sqrt{1-\frac{v_x^2+v_z^2}{c^2}}}\right).$$

The last expression reduces to $mg/\sqrt{1-v_x^2/c^2} = \gamma mg$ in the limit $v_z \to 0$. Thus, we can say that the balance reads a weight γmg, not mg!

Section #: 215 Your Name: ______________________________ 10 / 12 / 2012
Partners' name: ____

1. (a) Consider the simple pendulum. Its speed oscillates between zero at the ends and maximum at the center. Its velocity also changes direction periodically. Evidently its linear momentum is not conserved. Does this violate the principle of conservation of momentum ? Why or why not ?

(b) A bullet of mass 100 gm is fired at a pendulum bob of mass 400 gm which is stationary at its equilibrium position. The bullet hits the bob horizontally at a speed of 45 m/s and gets embedded in it. Use conservation of momentum and find out the common velocity of the bob and the bullet just after the inelastic collision. Assume that the rope from which the bob is suspended has negligible mass.

(c) The principle of conservation of momentum requires that there be no external forces. But in the last problem, there are external forces such as the tension in the rope, weights of the bob and the bullet. Can you justify why you still got the correct answer ?

(d) Next consider the same problem in part (b) with the only difference that the bullet hits the bob at a speed of $30\sqrt{2}$ m/s when the bob has just reached an extreme position while oscillating and the rope makes an angle of 45 degree with the vertical. Will the total momentum of the bob + the bullet be conserved in the collision ? Assume the rope cannot be stretched or broken.

(e) In the last problem, take the y axis along the rope (at the time of collision) and x axis perpendicular to it. Assume that the collision process occurs over a negligible time. Will the (i) x component (ii) y component of the total momentum be conserved ? Why or why not ?

(f) Find out the common velocity of the bob and the bullet just after the collision.

(g) If the collision process occurs over a 1 ms time interval, what is the average tension in the rope during the collision ? Clue: Impulse (area under force-time graph) gives the change in momentum.

If you have time, move on to problem 2.

2. (a) Consider the system shown in the figure. Suppose that the ball hits the pan at a speed v and thereafter moves with the pan. What is the common speed of the block / pan / ball just after the collision ? Assume that the pulley and rope are massless and that the collision process occurs over a negligible time. (Clue: Total impulse of all forces ON an object gives the net change in momentum of that object.)