$E = mc^2$: Einstein's derivation

(Dated: March 19, 2015)

All you need to know :

i) Kinetic energy of an object moving at a small speed v is $K = \frac{1}{2}mv^2$.
ii) Light comes in packets of energy : photons. A photon of wavelength λ has energy $E_{ph} = k/\lambda$, where k is a constant.
iii) Moving clocks run slow by a factor $\gamma = 1/\sqrt{1 - v^2/c^2}$.
iv) For small x (i.e., $x \ll 1$), $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2}$.

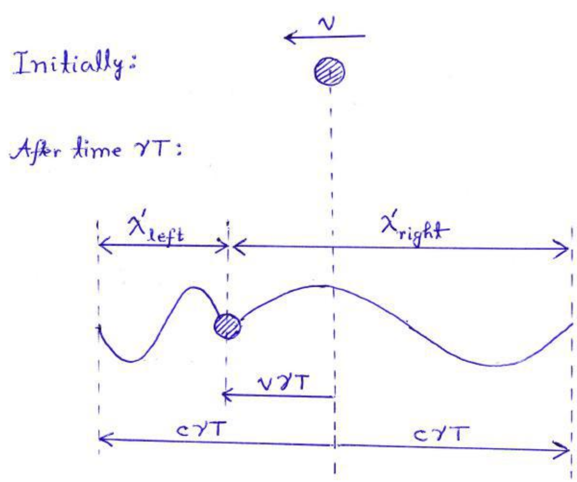
The derivation : Do the following thought experiment :

Consider an object at rest on the ground. It has a mass m and an energy E. Now think of viewing it from a train moving at a small speed v ($\ll c$). In this viewpoint, the object is moving at speed v and has a kinetic energy $K' = \frac{1}{2}mv^2$. So the total energy of the object in the train frame must be $E' = E + \frac{1}{2}mv^2$.

Now assume that, for some reason, the object emits two photons in opposite directions, and that in the ground frame, the two photons have the same wavelength λ. By symmetry, the object will stay at rest in the ground frame after the photon emission. But the photons carry energy. So the object will have lost some energy ΔE and some mass Δm. So the mass of the object after the light emission is $m - \Delta m$. And its energy is $E - \Delta E$. Since the two photons have the same wavelength λ in the ground frame, each of them carries the same energy, which is

$$\frac{\Delta E}{2} = \frac{k}{\lambda} \,. \tag{1}$$

Now look at the same phenomenon from the train frame. Here, the energy lost by the object is different from ΔE because the photons have different wavelengths! Let's see why. We need to invent a mechanism for light emission. Suppose, there is a charge on the object which oscillates at some rate. This results in light emission. Suppose in the ground frame, it completes one oscillation in time T, and we get one full light wave. Since light travels at speed c, it must have traveled a distance cT during this time. So, the wavelength in the ground frame is $\lambda = cT$. But what happens in the train frame? The charge is moving at speed v in the train frame. Since moving clocks run slow, more time has passed in the train frame during one oscillation! That time is γT. Now consider the figure below. During this time γT, light has traveled a distance $c\gamma T$ in both directions. But the charge on the object has also traveled a distance $v\gamma T$ to the left! So the light wave to the left has been compressed, and the light wave to the right has been expanded! In fact, the left photon has a wavelength $\lambda'_{left} = c\gamma T - v\gamma T = cT\gamma(1 - v/c) = \lambda\gamma(1 - v/c)$. And the right photon has a wavelength $\lambda'_{right} = c\gamma T + v\gamma T = \lambda\gamma(1 + v/c)$. This is the "Doppler shift".



Thus the energies of the two photons in the train frame are

$$E'_{left} = \frac{k}{\lambda\gamma(1 - v/c)} \qquad \text{and} \qquad E'_{right} = \frac{k}{\lambda\gamma(1 + v/c)} \,. \tag{2}$$

Therefore in the train frame, the object must have lost a total energy of

$$\Delta E' = E'_{left} + E'_{right} = \frac{k}{\lambda\gamma}\left[\frac{1}{1-v/c} + \frac{1}{1+v/c}\right] = \frac{2k}{\lambda\gamma}\frac{1}{1-v^2/c^2} . \tag{3}$$

When you substitute $\Delta E = 2k/\lambda$ from (1), and use $\gamma = 1/\sqrt{1-v^2/c^2}$, this becomes

$$\Delta E' = \Delta E \frac{1}{\sqrt{1-v^2/c^2}} . \tag{4}$$

Now, we know that originally, in the train frame, the object had an energy of $E + \frac{1}{2}mv^2$ (see the first paragraph). So, after the emission it must have the energy

$$E + \frac{1}{2}mv^2 - \Delta E' = E + \frac{1}{2}mv^2 - \Delta E \frac{1}{\sqrt{1-v^2/c^2}} . \tag{5}$$

We'll come back to this result shortly. Let us first make an argument similar to the first paragraph. After the emission, the object is still at rest in the ground frame where it has mass $m - \Delta m$ and $E - \Delta E$ (remember?). In the train frame, it is still traveling at speed v. So its total energy in the train frame must be

$$(E - \Delta E) + \frac{1}{2}(m - \Delta m)v^2 . \tag{6}$$

But (5) and (6) refer to the same quantity! So we can equate them :

$$E + \frac{1}{2}mv^2 - \Delta E \frac{1}{\sqrt{1-v^2/c^2}} = E - \Delta E + \frac{1}{2}(m - \Delta m)v^2 . \tag{7}$$

Canceling out equal terms, we get

$$-\Delta E \frac{1}{\sqrt{1-v^2/c^2}} = -\Delta E - \frac{1}{2}\Delta m v^2 . \tag{8}$$

But v is very small compared to c, i.e., $v/c \ll 1$. So we use the fact that for $x \ll 1$, $\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2}$. This gives

$$\Delta E\left[1 + \frac{1}{2}\frac{v^2}{c^2}\right] = \Delta E + \frac{1}{2}\Delta m v^2 , \qquad \text{or} \qquad \Delta m = \frac{\Delta E}{c^2} . \tag{9}$$

That's it! This means that losing an energy ΔE is equivalent to losing a mass of $\Delta E/c^2$. So, energy and mass are equivalent, and they are related by $m = E/c^2$!

---

[1] A. Einstein, "Does the inertia of a body depend on its energy content?," Annalen der Physik **18** (13): 639643; English translated version : Fourmilab site.

[2] E. Hecht, "How Einstein confirmed $E_0 = mc^2$," Am. J. Phys. **79** (6): 591-600.